# Concave/Convex functions and Maximum/Minimum Principles

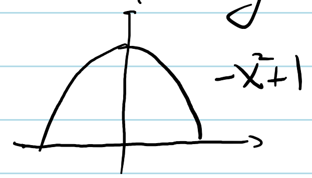
Feb, 21st, 2019.

- We will mainly focus on smooth function $f(x)$ on a closed interval $[a,b]$

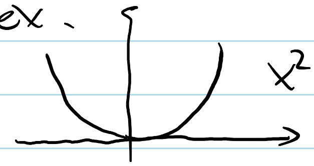
Def. If $-f''(x) \geq 0$, then $f$ is called concave. (or concave down)
in Chinese, 凹

If $-f''(x) \leq 0$, then $f$ is called convex (or concave up).
in Chinese, 凸

eg $f(x) = -x^2+1$. $-f''(x) = 2 > 0$ concave. (on any interval)




eg. $f(x) = x^2$, $-f''(x) = -2 < 0$ convex.




- We will mainly focus on concave (down) functions since if $f(x)$ is convex, then $-f(x)$ is concave.

- One property of concave function is that we can predict where its minimum valued will be obtained.

Prop. If $f(x)$ is concave on $[a,b]$, then the minimum of $f(x)$ cannot be obtained in the interior of the domain, $(a,b)$.

Cor. If $f(x)$ is concave on $[a,b]$ and $f(a)=f(b)=0$, then.
$f(x) \geq 0$ for all $x \in [a,b]$.

- This property is usually referred as to be Minimum Principle, which can be defined in a very wide sense.

Def. For a function $f(x)$ defined on a compact domain $D \subset \mathbb{R}^n$, if the minimum (maximum) cannot be obtained in the interior of the domain, then we say $f(x)$ satisfies the Minimum Principle (Maximum Principle).

Remark. People sometimes abuse the terminologies and refer both Min/Max Principle to Maximum Principle for simplicity. We will also call both Maximum Principle (Max-P) from now on.

- The previous discussion on concave function shows that concave function (on a closed interval) satisfies Max-P.

We also see that this is actually a property associated to the second order derivative (negative), which is a differential operator.

We are particularly interested in operators acting on functions on a compact domain $D \subset \mathbb{R}^n$ with zero boundary conditions.

Def. We say an operator $H$ has maximum principle (in domain $D$) if

$$Hf \geq 0 \text{ in } D \text{ implies } f \geq 0 \text{ in } D$$

for all $f$ with zero boundary condition (i.e., $f(x)=0$ for all $x \in \partial D =$ boundary of $D$)

Remark. This is a non-rigorous definition. The domain of the function space and the domain of the operator (to which functions can this operator be applied) have to be more precise in different context.

- Using this definition, we may express the Max-P of concave function as: the second order differential operator $H = -\frac{d^2}{dx^2}$ has Max-P on any closed interval $[a,b]$.

- The Max-P $\overset{\text{for } -\frac{d^2}{dx^2}}{\vee}$ can be extended to differential operator $H$ acting on (smooth) functions on $[a,b]$, given by
$$(Hf)(x) = a(x)\cdot\frac{d^2}{dx^2}f(x) + b(x)\cdot\frac{d}{dx}f(x) + V(x)\cdot f(x)$$
with appropriate assumptions on $a(x)$, $b(x)$, $V(x)$.

- The generalization of $\frac{d^2}{dx^2}$ on $\mathbb{R}^n$ $(n\geq 2)$ is called Laplacian operator, given by the sum of all second order partial derivatives: $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$

e.g. For $f(x_1, x_2)$ on $\mathbb{R}^2$, $\Delta f = \frac{\partial^2}{\partial x_1^2}f + \frac{\partial^2}{\partial x_2^2}f$

- The concept of concave function (convex) on $\mathbb{R}$ is generalized to be the so-called superharmonic (subharmonic) function on $\mathbb{R}^n$.

- In particular, for smooth functions,
  $f$ is called superharmonic if $-\Delta f \geq 0$
  subharmonic) if $-\Delta f \leq 0$

  $f$ is called harmonic if $f$ is both superharmonic and subharmonic (i.e. $\Delta f = 0$).

Notice that on $\mathbb{R}^1$, $\Delta f = f''(x) = 0 \Rightarrow f(x) = ax+b$
The only non-trivial harmonic function is linear function.
- On $\mathbb{R}^n$ $(n\geq 2)$, the harmonic functions are more complicated (and interesting)

- The Max-P is still true for $-\Delta$ on compact domain. But the proof is highly non-trivial on $\mathbb{R}^n$, $n\geq 2$.

- The phrase "harmonic" (in math) is frequently used in problems related to $\Delta$, (partial) second order derivative.

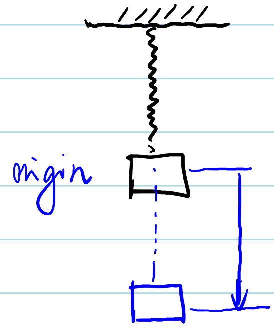
e.g. Harmonic Oscillator (of a Mass-spring system).

Newton's Second Law:

$$F = ma = f''(t)$$

Hooke's Law:

$$F = -kf$$




mass $m=1$

Hooke's constant $k>0$.

displacement function $f(t)$

accelleration $a(t) = f''(t)$.

F: force given by the spring

Putting two laws together,

$$f''(t) = -k \cdot f(t) \quad ( \Leftrightarrow \ -f''(t) = k \cdot f(t) ).$$

The solution is

$$f(t) = A\cos(\omega t + \varphi), \text{ for arbitary } A, \varphi.$$

$\omega$ (frequency) is given explicitly by $\omega = \sqrt{k}$

Notice that the equation $-f''(t) = k \cdot f(t)$ can be written as

$$Hf = k \cdot f \quad \text{by letting } H = -\frac{d^2}{dt^2},$$

which is the (generalized) eigen-value equation for $H$.

- The discrete version of $\Delta$ and maximum principle. (on $\mathbb{Z}^1$)

Recall the discrete version of the second order derivative $\frac{d^2}{dt^2}$ is the second order difference operator:

$$(\Delta X)_n =: X_{n+1} + X_{n-1} - 2X_n, \text{ for "function" } X: \mathbb{Z} \to \mathbb{R}.$$

- We are interested in eigenvalue problems and Max-P related to this discrete Laplacian. We also frequently drop the last term which simply shift the eigenvalue by 2.

If we consider a finite lattice $[1, 2, \cdots, n]$ and a vector $x = (x_1, \cdots, x_n)$,

the operator

$$(\Delta x)_n = x_{n+1} + x_{n-1}, \quad n = 1, \cdots, n$$

is properly defined if we extend the lattice $[1, \cdots, n]$ to $[0, 1, 2, \cdots, n, n+1]$ with the boundaries $j = 0$ and $j = n+1$.

If we impose the zero boundary condition on the extended vector $\tilde{x} = (x_0, x_1, \cdots, x_n, x_{n+1})$ such that

$$x_0 = x_{n+1} = 0,$$

then $\Delta$ will be consistent with the (free) Schrödinger matrix $H_0$ on $\mathbb{R}^n$

$$H_0 = \begin{bmatrix} 0 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{bmatrix}$$

- The goal is to study the maximum principle for the Schrödinger matrix $H = -\Delta + V := -\begin{bmatrix} 0 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix} + \begin{bmatrix} V_1 & & 0 \\ & \ddots & \\ 0 & & V_n \end{bmatrix}$

Lem. Assume that $V_j \geq 2$, $j = 1, \cdots, n$. For any vector $\vec{u} \in \mathbb{R}^n$,

(Max-P): if $H\vec{u} \geq 0$ (meaning all entries of $H\vec{u} \geq 0$), then $u_j \geq 0$ for all $j$.

(Strong Max-P): if $H\vec{u} > 0$ (at least one entry of $H\vec{u} > 0$), then $u_j > 0$ for all $j$.